The motion of rigid particles in a shear flow at low Reynolds number

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According to Jeffery (1923) the axis of an isolated rigid neutrally buoyant ellipsoid of revolution in a uniform simple shear at low Reynolds number moves in one of a family of closed periodic orbits, the centre of the particle moving with the velocity of the undisturbed fluid at that point. The present work is a theoretical investigation of how far the orbit of a particle of more general shape in a non-uniform shear in the presence of rigid boundaries may be expected to be qualitatively similar. Inertial and non-Newtonian effects are entirely neglected.

The orientation of the axis of *almost* any body of revolution is a periodic function of time in any unidirectional flow, and also in a Couette viscometer. This is also true if there is a gravitational force on the particle in the direction of the streamlines. There is no lateral drift. On the other hand, certain extreme shapes, including some bodies of revolution, will assume one of two orientations and migrate to the bounding surfaces or to the centre of the flow. In any constant slightly three-dimensional uniform shear any body of revolution will ultimately assume a preferred orientation.

1. Introduction

The motion under viscous forces of a rigid ellipsoid of revolution in a uniform simple shear at low Reynolds number was solved completely by Jeffery (1923) and has been accurately verified by the experiments of Trevelyan & Mason (1951). The centre of such an ellipsoid moves with the velocity of the undisturbed fluid at that point, and in a co-ordinate frame moving with the centre the ends of its axis describe a closed spherical ellipse. There is an infinite family of such orbits, and that actually described depends on the initial orientation of the particle. For a long ellipsoidal rod of large axis ratio the axis is, for most of the time, aligned almost parallel to the streamlines, but then reverses itself periodically. Jeffery postulated that in practice the particle would assume the orbit corresponding to minimum mean dissipation of energy.

Taylor (1923) found experimentally that an ellipsoid of revolution in a uniform shear did not move in a quite closed orbit, but that there was a slow drift through the continuous family of orbits calculated by Jeffery, until the ellipsoid was rotating with its axis in a plane perpendicular to the vortex lines. It has been suggested by Saffman (1956) that this might be due to non-Newtonian properties of the suspending medium. Mason & Manley (1956) also found a slow drift, but the evidence is inconclusive.

Saffman (1956) remarked that the linearity of the Stokes equations for slow flow implies that a slight non-uniformity of the shear, or the presence of boundary walls, could not explain this drift. A more developed version of this argument is presented in § 2 of this paper, where it is shown that it may also be applied to other cases of interest.

Except in § 6, all the flows in which particles are here considered are unidirectional, so that locally the velocity field is approximately a simple shear. If the shear is non-uniform the magnitude and direction of the transverse velocity gradient may vary across the flow. Except for local disturbances caused by irregularities in the boundary surfaces this definition covers fully developed laminar flow through pipes and channels of uniform cross-section and, with slight modification, that in a Couette viscometer. In § 6 is considered the motion of an extremely small particle in a flow which is not locally a simple shear but which is essentially three-dimensional, for example that in a steadily diverging pipe.

The motion of a dilute suspension of small neutrally buoyant spheres in the Poiseuille flow in a tube of circular cross-section has been studied by Segré & Silberberg (1961, 1962). They found that the spheres slowly migrate laterally until they are concentrated in an annular region distant 0·6 tube radii from the axis. This appears to be a property of an individual sphere in such a flow, independently of the presence of the others, and the rate of migration agrees with that expected from dimensional considerations to arise from the inertia of the fluid. The Reynolds number based on tube diameter was of order 30, whereas that based on particle size was less than unity. Such a migration across the streamlines, though to the centre of the tube, had previously been suggested by several writers (e.g. Starkey 1956; Scott-Blair 1958). Goldsmith & Mason (1961*a, b*) found no migration of rigid spheres, disks or rods in Poiseuille flow even very close to a wall. In this case the Reynolds number based even on tube diameter was small compared to unity. Flexible rods, and liquid drops, on the other hand, always migrated towards the tube axis. A similar effect has been observed by Christopherson & Dowson (1959) when a large, heavy sphere falls slowly through a stationary viscous liquid in a vertical tube; they calculated the flow in the narrow region between the sphere and the tube wall using a lubrication approximation, and showed that it would fall vertically whatever the distance of its centre from the tube axis. An eccentric position corresponds to a maximum rate of fall, and they found that whatever the initial conditions the sphere migrated laterally until it assumed this position.

The same negative arguments that Saffman used about the effect of tube walls, non-uniform shear, etc., will be shown to be applicable here. The explanation for this migration is not to be found within the context of the Stokes equations—it must be due to non-linear effects of some kind, inertial or non-Newtonian forces.

Most of these results apply to spheres or ellipsoids of revolution. Goldsmith & Mason (1961*b*), however, found no migration even for rigid disks and rods of circular cross-section, so the questions arise: If the particle is rigid but of general shape, does the same qualitative picture hold? In the limit of zero Reynolds number are the orbits still closed? In a non-uniform shear might a long thin unsymmetrical particle spend more time with one end aligned upstream rather than downstream? These questions are viewed within the framework of the Stokes approximation to the equations of motion, and in the absence of external forces acting on the particle, and assuming it is strictly rigid, so any results are independent of the Reynolds number of the motion, provided only it is very much less than unity. It is found that there exist shapes of body for which migration and preferential orientation do take place, and also that the orientation of a particle does not always describe a closed orbit, relative to axes moving with its centre. These effects, when present, will dominate over those due to the inertia of the fluid or to small external disturbances, but they are absent from the shapes on which accurate experiments have actually been performed. It is shown that for almost any body of revolution (including particles of the shape of the disks and rods used by Goldsmith & Mason), the orbit will be accurately closed, and no migration will occur in any unidirectional flow. The same result usually holds if there is an external force on the particle acting parallel to the tube axis. This occurs when there is incomplete density matching between the particle and the suspending medium and the tube is mounted vertically.

If the Reynolds number based on the tube diameter is appreciable, the motion of a sufficiently small particle will still be determined primarily by the Stokes equations of motion in terms of the local rate of shear. This is investigated in §§ 3–5. The inertia forces are negligible because they are proportional to the square of the particle size. If lateral migration is predicted even in the absence of inertia, and there are rigid bodies for which this is so, it will dominate over effects due to the residual inertial terms. This has not yet been observed experimentally. For spheres, however, migration cannot arise in this way, and that observed by Segré & Silberberg (1961) must be accounted for by invoking the neglected inertia terms or non-Newtonian effects. That is outside the scope of this paper. When the overall Reynolds number is also small compared to unity, on the other hand, the prediction of no migration for most rigid bodies of revolution is well verified experimentally (Goldsmith & Mason 1961*a*, *b*). For flexible particles and liquid drops migration occurs and is presumably associated with their elastic properties.

For rigid particles it is possible to deduce from the Stokes equations some answers by quite general arguments. § 2 will be devoted to a detailed statement of Saffman's argument for non-uniform unidirectional flows. In § 3 the context will be specialized to uniform simple shear, to discuss the limiting orbit as the particle size tends to zero, and the equations of motion are deduced for general bodies. In §§ 4 and 5 some illustrative examples are given of special shapes which do exhibit migration. In § 6 a solution is given for the orientation of a general body of revolution in a three-dimensional uniform time-independent shear. Unfortunately when the rate of strain tensor in the neighbourhood of the particle is varying with time the solution cannot be determined explicitly except for one special case and this restricts the usefulness of the result. The special case describes the orientation of an infinitesimal particle in a general (non-unidirectional) viscous flow, provided the structure of the rate of strain tensor is effectively constant along the streamline along which the particle moves. The magnitude of the rate of strain may vary, but a simple re-definition of the parameter describing time reduces the problem to the case when the magnitude is constant. The particle describes the same orbit, though at a different rate. There is either an asymptotic orientation of the axis of the particle or an asymptotic plane in which it rotates. This contrasts sharply with the orbits calculated by Jeffery (1923) for a simple shear. From this standpoint simple shear is a very special, though important, case, and any slight three-dimensionality in a unidirectional flow which is maintained along a streamline completely alters the character of the particle orbits. The arguments of this paper are necessarily somewhat disjointed, and for convenience the main train of reasoning is restated in § 7, together with the principal conclusions.

2. The mirror-symmetry time-reversal theorem

2.1. The following result holds for general bodies in any unidirectional flow: *When moving in a steady unidirectional shear flow at small Reynolds number under the action of viscous forces alone, to every orbit of a given finite rigid body there corresponds one of the body of opposite mirror-symmetry. The corresponding orbits are 'mirror images' obtained by reflexion in a plane perpendicular to the streamlines, but are traversed in opposite senses.* This has probably been known in essence for some time, but the author has not seen it explicitly in print, and accordingly a proof is given here.

It follows very simply from the general principle of invariance under changes of mirror-symmetry, and the linearity of the Stokes equations. At low Reynolds number the Stokes approximation gives the velocity field near a finite body, and, though it is not necessarily uniformly valid, it can be used to calculate the forces on the body. The full Navier–Stokes equations are invariant under the reflexion of velocities, forces, and co-ordinates in (say) the (x_2, x_3)-plane, provided the pressure is left unaltered. The Stokes equations have the additional property that under time reversal, i.e. reversal of all velocities and forces, they are still satisfied, and if the resultant forces and couples on the body vanished, they still vanish. But reflexion in the (x_2, x_3)-plane and velocity reversal together leave unaltered a velocity field at large distances which is parallel to Ox_1 and independent of x_1. The relative positions of points on the body become those of the body of mirror-symmetry, and the orbit of each becomes its time-reversed mirror image.

Stated analytically, the argument considers a body, whose surface is made up of points P, with Cartesian co-ordinates $r_\alpha(P, t)$, moving in a velocity field u_α so that

$$\frac{\partial p}{\partial x_\alpha} = \eta \frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}, \quad \frac{\partial u_\alpha}{\partial x_\alpha} = 0.$$

At the surface of the body

$$\frac{dr_\alpha}{dt} = u_\alpha,$$

and at large distances from it

$$u_1 \to U(x_2, x_3); \quad u_2, u_3 \to 0.$$

The resultant force and couple on it are

$$F_\alpha = \int \left\{ -p\delta_{\alpha\beta} + \eta \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right) \right\} dS_\beta = 0,$$

and
$$G_\alpha = \int \epsilon_{\alpha\beta\gamma} r_\beta \left\{ -p\delta_{\gamma\delta} + \eta \left(\frac{\partial u_\gamma}{\partial x_\delta} + \frac{\partial u_\delta}{\partial x_\gamma} \right) \right\} dS_\delta = 0.$$

Consider the body whose surface is made up of points P', with co-ordinates $r_\alpha(P',t) = \Gamma r_\alpha(P, -t)$, where

$$\Gamma r_1 = -r_1, \quad \Gamma r_2 = r_2, \quad \Gamma r_3 = r_3,$$

moving in a velocity field $u'_\alpha(P',t) = -\Gamma u_\alpha(P, -t)$ with pressure

$$p'(P',t) = -p(P, -t).$$

Then it is easily verified that all the above equations of motion are satisfied also by the primed variables, in particular

$$u'_1 \to U(x_2, x_3), \quad u'_2, u'_3 \to 0,$$

at large distances from the body. But the co-ordinates

$$r_\alpha(P',t) = \int u'_\alpha\, dt = \int -\Gamma u_\alpha(P, -t)\, dt = \Gamma r_\alpha(P, -t)$$

describe the position at successive times of the configuration $\{P'\}$ of opposite mirror-symmetry to $\{P\}$ and show that it describes the image orbit in the reverse sense.

2.2. This theorem is of particular application to bodies of revolution, for these have the property that when reflected in any plane containing their axis they coincide with their image. If, therefore, at any moment in the orbit of such a particle the axis lies in a plane perpendicular to the streamlines, the previous and subsequent parts of the orbit must be mirror images in this plane, for these two parts have this one moment in common. If, at any subsequent moment, the axis again lies perpendicular to the streamlines the orbit can also be divided into two parts, mirror images in a second plane parallel to the first. But this means that the whole orbit can be generated by successive reflexions in two parallel planes, and it must be accurately periodic, with period twice the time taken to traverse that part which lies between them.

This statement is independent of the variations of velocity at right angles to the undisturbed streamlines, even in the presence of rigid walls parallel to the streamlines. Thus, provided only that there is a basic reversing motion, so that the axis twice lies at right angles to the basic flow, a complete answer can be given to the questions posed in the introduction for any neutrally buoyant axisymmetric body in any parallel flow. Taken in a co-ordinate frame moving with suitable velocity the orbits are closed, there is no more time spent pointing upstream rather than downstream, and any tendency to drift across the streamlines in one part of the orbit is accurately cancelled by a reverse tendency in another part. This has been verified by Goldsmith & Mason (1961*b*) for rigid spheres and rods in the neighbourhood of a wall.

2.3. This theorem may be extended in two ways. If there is a constant external force acting on the body (e.g. gravity), *in the direction of the basic streamlines*, exactly the same statements hold. For under co-ordinate reflexion in the (x_2, x_3)-plane, the x_1-component only of all forces on the body is reversed. Under time reversal all force components change their sign so under both transformations the x_1-component is unaffected. If the resultant force is in this direction, and if there is no resultant couple about it, all the consequences of a periodic orbit follow. This rules out an explanation of Christopherson & Dowson's results by the Stokes equations, for the resultant force on the sphere was parallel to the tube axis, so that lateral motions are excluded by this theorem, provided the walls of their tube can be taken as accurately vertical. If gravity does not act along the undisturbed streamlines the above theorem is not true. An instance of this has been given by Bretherton & Rothschild (1961). Thus in the usual arrangement of a Couette viscometer particles must be accurately matched in density to the surrounding liquid if gravity is not to introduce important qualitative effects.

Secondly, the theorem is also easily proved when the undisturbed streamlines are not strictly unidirectional, but can be cast in terms of orthogonal co-ordinates $(\lambda_1, \lambda_2, \lambda_3)$ as the curves $\lambda_2, \lambda_3 =$ const., where the element of length corresponding to infinitesimal changes $\delta\lambda_1, \delta\lambda_2, \delta\lambda_3$ is independent of λ_1. The most important case is when the streamlines are circles with centres on an axis of symmetry, as in a Couette viscometer. The basic flow and geometry is then independent of θ, the angular displacement about the axis, and reflexion corresponds to changing the sign of θ.

Finally it should be noticed that the theorem has only been proved if the bounding surfaces of the basic flow are rigid, and stationary. In particular cases extensions to free or moving surfaces may be made, but each case must be examined on its merits. If more than one particle is present the theorem may be applied only when there is axial symmetry of the particle configuration as a whole, e.g. two interacting spheres which are otherwise isolated.

2.4. This theorem shows that the orbit of the axis of a body of revolution is strictly periodic, and that lateral migration will not occur, provided only that its axis lies at least twice in a plane perpendicular to the streamlines. The nub of the argument is the existence of the basic reversing mechanism. This might cease very close to a wall, or in the central region of a parabolic profile but for sufficiently small particles it will be present in the remainder of the flow if it is present in a uniform shear in the absence of walls. In the limit as the particle size tends to zero, the broad features of the particle orbit must become dominated by the local flow pattern, which is a uniform simple shear. Reversals certainly took place for the experiments mentioned in the introduction, for Taylor and Manley & Mason examined them explicitly, and for a sphere any diameter is an axis of symmetry. In the next section it will be shown that the motion of almost all bodies of revolution in a uniform simple shear is the same, so far as orientation is concerned, as that of some ellipsoid, for which reversals certainly occur. However, there are certain rather extreme shapes, including some bodies of revolution, in which the particle moves directly into one of two opposite orientations, and remains there, migrating steadily across the streamlines. A small particle in the parabolic velocity profile in a pipe or between two parallel plates will migrate either to the centre or to the walls. In these regions the shear cannot be regarded as even approximately uniform, and migration will either be halted or reversed. If it is merely halted in either of these regions, the result will be an ultimate concentration of particles there. Prediction of their detailed behaviour under these circumstances is not attempted, and these suggestions remain speculation. The overturning motion might also be halted everywhere in a vertical tube if there were a density difference between the particle and the suspending medium and the particle were asymmetric between its two ends. Differential forces associated with the sedimentation of the particle as a whole might be adequate to overcome the effects of the shear. It is clear, however, that when migration occurs, even in a uniform shear, it is likely to prove of some importance, so in the next three sections attention is concentrated on neutrally buoyant particles in a uniform shear away from rigid walls, so that their motion (except for translation down the streamlines) depends only on their orientation, and is independent of spatial position.

3. The equations of motion in a uniform shear; particle characterization

3.1. A general rigid particle is moving under no external forces in a motion compounded of a straining motion and a rotation, the rate of strain tensor and the vorticity being independent of position. Because of the linearity of the Stokes equations its rate of rotation ω_α can be expressed as the sum of the rotation of the fluid Ω_α (half the vorticity) and a term depending linearly on the rate of strain, $E_{\beta\alpha}$,

$$\omega_\alpha = \Omega_\alpha + \tfrac{1}{2}B_{\alpha\beta\gamma}E_{\beta\gamma}.$$

Throughout what follows summation over the range 1, 2, 3 is tacitly assumed for every repeated Greek suffix, whereas for a Roman suffix it will always be explicitly indicated where required. The above expression, which is with reference to axes instantaneously coinciding with directions fixed in the body, is a tensor one, and does not depend for its truth on any particular choice of the rate of strain tensor $E_{\beta\gamma}$. $B_{\alpha\beta\gamma}$ is therefore a Cartesian tensor of rank three, which may be regarded as symmetric between the suffices β and γ, and which is a property of the body alone. In particular consider the simple shear $\mathbf{u} = G(\mathbf{r}\,.\,\mathbf{n})\,\mathbf{l}$ where $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, are a right-handed triad of unit vectors, $\mathbf{l}$ being along the streamlines, $\mathbf{m}$ along the vortex lines, and G is a constant. Then

$$\omega_\alpha = \tfrac{1}{4}G\{\epsilon_{\alpha\beta\gamma}(n_\beta l_\gamma - n_\gamma l_\beta) + B_{\alpha\beta\gamma}(n_\beta l_\gamma + n_\gamma l_\beta)\}. \tag{1}$$

3.2. Certain facts about $B_{\alpha\beta\gamma}$ can be deduced at once from considerations of symmetry. If a body is invariant under rotations about Ox_1 through a right angle, the transformation matrix being

$$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & -1 & \cdot \end{pmatrix},$$

then the corresponding $B_{\alpha\beta\gamma}$ must be connected by the following relations

$$\begin{aligned} &B_{112} = B_{113} = -B_{112}, \quad B_{123} = -B_{132}, \quad B_{122} = B_{133}, \\ &B_{211} = B_{311} = -B_{211}, \quad B_{212} = B_{313}, \quad B_{213} = -B_{312}, \\ &B_{222} = B_{333} = -B_{222}, \quad B_{233} = -B_{332} = -B_{233}. \end{aligned}$$

The motion is thus determined by four independent constants, B_{111}, B_{122}, B_{212}, B_{213}. Under co-ordinate transformations which involve a change of reflexion symmetry, e.g. that with matrix

$$\begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix},$$

angular velocity is an axial vector (the first and third components change sign) and if the body is also invariant under this transformation the tensor character of equation (1) demands that

$$B_{111} = B_{122} = B_{212} = 0.$$

If, therefore, a body is unaltered by rotation through a right angle about an axis, and by reflexion in a plane containing that axis, the number of constants which describe its motion is reduced to one. An important special case of this class is a body of revolution. Note that there is no condition on symmetry about a plane perpendicular to the axis. With axes fixed in space, but chosen so that Ox_1 instantaneously coincides with the axis of the body, and Ox_1x_2 with a plane of reflexion symmetry, the equations of motion reduce to

$$\left.\begin{aligned} \omega_1 &= \tfrac{1}{2}G\{n_2 l_3 - n_3 l_2\}, \\ \omega_2 &= \tfrac{1}{2}G\{(1+B)\, n_3 l_1 - (1-B)\, n_1 l_3\}, \\ \omega_3 &= \tfrac{1}{2}G\{(1-B)\, n_1 l_2 - (1+B)\, n_2 l_1\}, \end{aligned}\right\} \tag{2}$$

and provided $|B| < 1$, these are exactly those given by Jeffery for an ellipsoid of revolution with axes in the ratio

$$\{(1-B)/(1+B)\}^{\frac{1}{2}} : 1 : 1.$$

It will be shown (§ 5.3) that, for some very long bodies, $|B|$ may be greater than unity, but with this proviso there follows the general result:

With the exception of certain very long ones, the motion in a uniform simple shear of a rigid body of revolution is mathematically identical, at least in so far as rotation is concerned, with that of some ellipsoid of revolution.

This has been confirmed for long circular cylinders in a uniform shear by Trevelyan & Mason (1951).

3.3. One further class of specially symmetric bodies is worth considering, those which are invariant under separate reflexions in two perpendicular planes. This includes members of the previous class as a special case, when the body is also unaltered when one plane is rotated into the other. The equations of motion then reduce to

$$\left.\begin{aligned} \omega_1 &= \tfrac{1}{2}G\{(1+B_1)\, n_2 l_3 - (1-B_1)\, n_3 l_2\}, \\ \omega_2 &= \tfrac{1}{2}G\{(1+B_2)\, n_3 l_1 - (1-B_2)\, n_1 l_3\}, \\ \omega_3 &= \tfrac{1}{2}G\{(1+B_3)\, n_1 l_2 - (1-B_3)\, n_2 l_1\}, \end{aligned}\right\} \tag{3}$$

where B_1, B_2, B_3 are three independent arbitrary constants. Reflexion symmetry about a third plane at right angles to the other two does not place any further restrictions on these constants. For an ellipsoid of axes a, b, c,

$$B_1 = \frac{b^2-c^2}{b^2+c^2}, \quad B_2 = \frac{c^2-a^2}{c^2+a^2}, \quad B_3 = \frac{a^2-b^2}{a^2+b^2},$$

implying the relation

$$B_1 B_2 B_3 + B_1 + B_2 + B_3 = 0.$$

In general, therefore, the motion of bodies invariant under reflexion in two, or three, axes is *not* the same as that of some ellipsoid.

3.4. There remains the question of whether there exists a body for which the 18 coefficients $B_{\alpha\beta\gamma}$ can assume arbitrarily assigned values. Three may be fixed by choosing suitable axes, but the other 15 are determined by the geometry of the body. To throw further light on this it is worth considering a model which is somewhat artificial, yet could in principle be realized to arbitrary accuracy, for which the equations of motion can be obtained fairly easily, and which retains sufficient generality to be expected to show qualitatively the main features of actual bodies. An assembly of N small ellipsoids are rigidly connected by long rods of negligible resistance. The forces on the ellipsoids are given by the difference between the velocities of their centres and the local velocity of the undisturbed fluid, and if the connecting rods are sufficiently long these forces can be calculated without reference to the presence of the remainder of the assembly. The couples on the individual ellipsoids are negligible, provided none of the overall dimensions of the assembly are too small. The requirements that the relative separations and orientations should remain unchanged, and that the resultant force and couple on the assembly should vanish, yield the equations of motion.

The derivation of these is straightforward but adds nothing to the argument presented here, so is omitted. The results assume a simple form when all the ellipsoids are spheres, at the points $P^i (i = 1, \ldots, N)$. The force acting at P^i is then a positive constant, A^i, say, times the local relative velocity between the assembly and the fluid. If r^i_β is the position vector of P^i relative to a point P^0 fixed in the body, P^0 can be chosen so that

$$\sum_{i=1}^{N} A^i r^i_\beta = 0.$$

P^0 is thus the 'centroid' of the spheres, and if C_2+C_3, C_3+C_1, C_1+C_2 are their 'principal moments of inertia', i.e. axes are chosen so that

$$\sum_i A^i r^i_2 r^i_3 = \sum_i A^i r^i_3 r^i_1 = \sum_i A^i r^i_1 r^i_2 = 0,$$

$$C_1 = \sum_i A^i r^{i2}_1, \quad C_2 = \sum_i A^i r^{i2}_2, \quad C_3 = \sum_i A^i r^{i2}_3; \quad C_1, C_2, C_3 > 0,$$

the equations of motion are

$$(C_2+C_3)\,\omega_1 = \tfrac{1}{2}G\{C_2 n_2 l_3 - C_3 n_3 l_2\},$$
$$(C_3+C_1)\,\omega_2 = \tfrac{1}{2}G\{C_3 n_3 l_1 - C_1 n_1 l_3\},$$
$$(C_1+C_2)\,\omega_3 = \tfrac{1}{2}G\{C_1 n_1 l_2 - C_2 n_2 l_1\},$$

which are identical with those describing the motion of a real ellipsoid, of semi-axes in the ratio $\sqrt{C_1}:\sqrt{C_2}:\sqrt{C_3}$. This model is thus too restricted for our purpose, and it is necessary to consider ellipsoids, instead of spheres, at the points P^i.

The force on a single ellipsoid in a uniform stream is given by Lamb (1932, p. 605). This depends on the orientation, being, for a flat circular disk edge on, only $\frac{2}{3}$ of that when broadside on. The resistance at P^i must be described by a positive-definite symmetric second-rank tensor, whose components relative to axes fixed in the assembly are constant. This would be true even if the ellipsoid there were replaced by a body of general shape of size very small compared to the separation between the points P^i. The 'centroid' of such an assembly of resistances is meaningless, and although the analysis can be carried through quite straightforwardly it is difficult to see what the general form of the tensor $B_{\alpha\beta\gamma}$ is. However, departures from isotropy of the resistance of bodies for which they have been calculated are small, the largest being for a long thin ellipsoid of revolution when the lateral resistance is twice the longitudinal. This is an extreme case, so it is not unreasonable to write the resistance tensor

$$A^i_{\beta\gamma} = A^i\delta_{\beta\gamma} + \alpha a^i_{\beta\gamma}$$

and to restrict attention to the case when α is small. A^i is positive, but $a^i_{\beta\gamma} = a^i_{\gamma\beta}$ is arbitrary.

Again taking P^0 as the 'centroid' and the axes as the 'principal axes' of the isotropic part of the $A^i_{\beta\gamma}$ the equations of motion become, to first order in α,

$$\omega_\alpha = \tfrac{1}{4}G\sum_{r,s}\left\{\epsilon_{\alpha rs}(n_r l_s - n_s l_r) + \epsilon_{\alpha rs}\frac{C_r - C_s}{C_r + C_s}(n_r l_s + n_s l_r) \right.$$
$$\left. + \alpha\sum_{u,v=1}^{3}\sum_{i=1}^{N} 4\epsilon_{\alpha uv}\frac{r^i_u a^i_{vs} r^i_r}{C_u + C_v}\frac{C_r}{C_r + C_s}(n_r l_s + n_s l_r)\right\}, \tag{4}$$

where $\epsilon_{\alpha\beta\gamma}$ is the isotropic antisymmetric tensor of rank three. For given r^i_u and A^i there is no restriction on the anisotropic part of the resistance at P^i, other than that it should be symmetric. The core

$$\sum_i r^i_u a^i_{vs} r^i_r$$

of the last term of equation (4) can have any value symmetric between the suffices u, r and v, s. It is thus apparent that the general form of the tensor describing the motion of a body in a uniform shear that is consistent with this model is

$$B_{\alpha rs} = \epsilon_{\alpha rs}\frac{C_r - C_s}{C_r + C_s} + b_{\alpha rs}, \quad C_r > 0,$$

where $b_{\alpha rs}$ is arbitrary, but in some sense small.

Though departures of $B_{\alpha\beta\gamma}$ from that for an ellipsoid are small they may, in certain circumstances, exert a crucial influence on the orbits of such particles, leading to qualitatively quite different behaviour. The equations of motion (equation (1)) are non-linear in the direction cosines l_α, m_α, n_α, and except for a slight generalization given in § 6 of the results given by Jeffery (1923) for ellipsoids of revolution, the author has been unable to find any interesting analytic solutions. However, the arrays of ellipsoids considered in this section, though very unrealistic models of any particles likely to be encountered in practice, can assist our understanding of some of the qualitative effects of non-zero $b_{\alpha rs}$. In the next section attention is turned to one particular array, to illustrate this.

4. An asymmetric long thin particle yielding equilibrium orientations

4.1. A characteristic of the equations of motion found by Jeffery for ellipsoids of finite axis-ratios in a uniform simple shear (equation (3)) is that there is no orientation in which the particle ceases to rotate. The constants B_1, B_2, B_3 are all less than unity, and there is no real position of equilibrium. This, however, is not true for more general shapes of body. Consider, for example, the particle illustrated in figure 1. An array of four rigidly connected ellipsoids form a square of

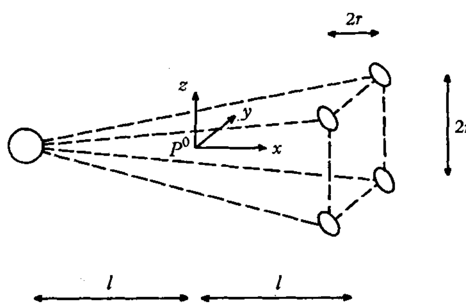

FIGURE 1. An array of ellipsoids which may remain in an equilibrium orientation in a shear flow.

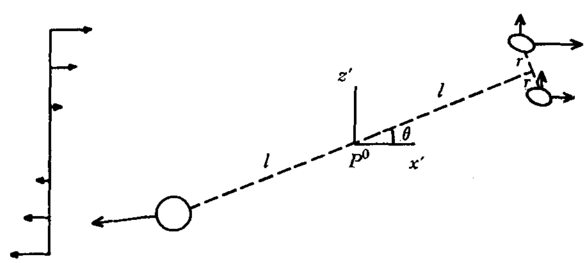

FIGURE 2. The forces on the array when in an equilibrium orientation. The array is moving as a whole across the streamlines of the basic shear.

side $2r$, while distant $2l$ from the centre of the square, and normal to its plane, is a sphere. The axes of the ellipsoid do not coincide with the principal axes of the array, and their resistance will be taken as

$$A\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + \alpha A \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}.$$

The resistance of the sphere will be taken as $4A$.

The constants C_1, C_2, C_3 are seen to be $8Al^2$, $4Ar^2$, $4Ar^2$ respectively, and if $\alpha = 0$ the array will move like an ellipsoid of revolution with axes in the ratio $\sqrt{2l} : r : r$. If $\alpha > 0$, however, a new element enters the situation. If there is a velocity difference between the ellipsoids and the local fluid in the direction Ox_1, there will be a transverse force in the x_3-direction acting on them. If the array is long and thin, i.e. r/l is small, such a velocity difference must exist whenever the long axis is inclined to the (x_1', x_2')-plane (figure 2), for then the sphere and ellipsoids are in regions of substantially different fluid velocity, but their separation must remain unchanged. This transverse force can exert a comparatively large moment about the centre of the long axis and this can halt the overturning characteristic of the array which would dominate if $\alpha = 0$.

4.2. To see this we consider the equations of motion. Writing $r^2/2l^2 = p$, it is straightforward to verify that equation (4) becomes

$$\omega_1 = \tfrac{1}{2}G\left\{n_2 l_3 - n_3 l_2 + \frac{\alpha}{1+p}(n_2 l_1 + n_1 l_2)\right\},$$

$$\omega_2 = \tfrac{1}{2}G\left\{n_3 l_1 - n_1 l_3 - \frac{1-p}{1+p}(n_3 l_1 + n_1 l_3) - \frac{\alpha}{2(1+p)}2n_1 l_1 + \frac{p\alpha}{1+p}2n_3 l_3\right\},$$

$$\omega_3 = \tfrac{1}{2}G\left\{n_1 l_2 - n_2 l_1 + \frac{1-p}{1+p}(n_1 l_2 + n_2 l_1) - \frac{p\alpha}{1+p}(n_2 l_3 + n_3 l_2)\right\}.$$

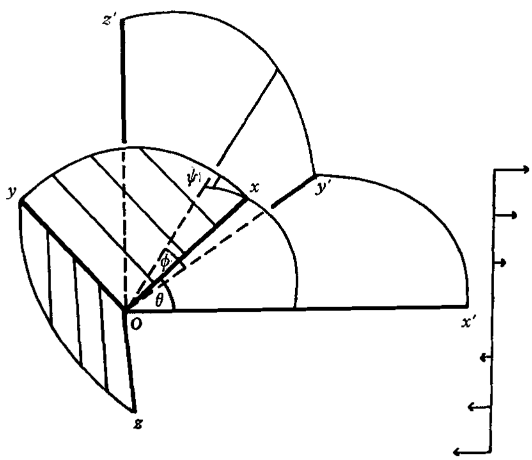

FIGURE 3. The Euler angles (θ, ϕ, ψ) defining the relative orientations of the axes $Oxyz$ of the body and the axes $Ox'y'z'$ fixed in space. The undisturbed velocity field is

$$u' = Gy'.$$

There is an orientation of the array in which the Euler angles defined in figure 3 are

$$\theta = \theta_0, \quad \phi = \tfrac{1}{2}\pi, \quad \psi = -\tfrac{1}{2}\pi,$$

and

$$(l_1, l_2, l_3) = (\cos\theta_0, 0, -\sin\theta_0),$$

$$(m_1, n_2, m_3) = (0, 1, 0),$$

$$(n_1, n_2, n_3) = (\sin\theta_0, 0, \cos\theta_0),$$

and the angular velocity vanishes. This is when θ_0 satisfies

$$1 - \frac{1-p}{1+p}\cos 2\theta_0 - \frac{\alpha(1+2p)}{2(1+p)}\sin 2\theta_0 = 0,$$

which has two real roots in $0 < \theta_0 < \pi$ if

$$\frac{p}{(1+2p)^2} < \frac{\alpha^2}{16}.$$

Thus provided the array is sufficiently long there are at least two orientations in which rotation will cease. If the axis of the array is reversed there are two more equilibrium orientations at

$$(\theta, \phi, \psi) = (\pi - \theta_0, -\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$$

and four more at the 'mirror-image' positions

$$(\pi - \theta_0, \tfrac{1}{2}\pi, \tfrac{1}{2}\pi), \quad (\theta_0, -\tfrac{1}{2}\pi, \tfrac{1}{2}\pi).$$

More detailed analysis reveals that none of these eight equilibrium orientations is completely stable to infinitesimal disturbances, i.e. if the orientation of the particle is altered slightly it will move further away from equilibrium. However, in the next section we will give an example of a body which not merely has orientations for which the rotation ceases, but which moves directly into one of them from almost any initial position, and remains there.

The forces on the array in an equilibrium orientation are illustrated in figure 2. It should be noted that the whole array is moving sideways across the undisturbed shear at a rate which to first order in α is $\frac{1}{2}Gl\alpha \sin\theta_0$, so if the particle were to remain in the equilibrium position it would migrate steadily across the undisturbed streamlines.

5. Bodies of revolution in a simple shear

5.1. In the previous section the alignment of the particles in an equilibrium orientation depended on a transverse force arising from the asymmetry of the particle about its axis. This asymmetry is not necessary, however, to produce such behaviour. It also arises in bodies of revolution. In § 3 it was shown that the orientation of a body of revolution is governed by equation (2) which involves one arbitrary constant B. If $|B| < 1$ the orbit (in terms of the Euler angles defined in figure 3) is

$$\cot^2\theta = p^2(1-k^2)\Big/\left\{\cot^2\left(\frac{Gpt}{p^2+1}\right)+k^2\right\}, \quad \cos^2\phi = k^2(1+p^{-2}\cot^2\theta),$$

where $p^2 = (1+B)/(1-B)$ and k^2 is a constant of integration. This result is taken with a change of notation from Jeffery (1923). If $|B| > 1$, however, it must be replaced by

$$\cot^2\theta = p'^2(1-k^2)\Big/\left\{\coth^2\left(\frac{Gp't}{1-p'^2}\right)-k^2\right\}, \quad \cos^2\phi = k^2\{1-(p')^{-2}\cot^2\theta\},$$

where $p'^2 = (B+1)/(B-1)$. This is no longer periodic, and as $Gt \to \infty$ the axis tends to a position $\phi = \pm\frac{1}{2}\pi$, $\cot\theta = \pm p'$, which is at right angles to the vortex lines, making an angle $\pm \cot^{-1}p'$ with the streamlines. Thus, there are two stable equilibrium orientations, and as $Gt \to -\infty$ two more unstable ones are found in the image positions on reflexion in a plane perpendicular to the streamlines. The result of § 2 that the orbits are periodic breaks down in this case because of the failure of the basic reversal mechanism.

5.2. It is not possible to construct a strict body of revolution from an assembly of several ellipsoids, unless they are all aligned along the axis. But for any body which has reflexion symmetry about two perpendicular planes and is invariant when one such plane is rotated through a right angle into the other, the tensor $B_{\alpha\beta\gamma}$ depends on only one constant B, and the equations for the rotation of the body are formally identical with those for a body of revolution.

We consider the array consisting of a sphere at the point $(-l, 0, 0)$ of resistance $4A$, and ellipsoids at the points forming a square of side $2r$,

$$(l, r, r); \quad (l, r, -r); \quad (l, -r, r); \quad (l, -r, -r),$$

with resistances which have an isotropic part of magnitude A, and an anisotropic part described by the tensors

$$\alpha\begin{pmatrix} . & 1 & 1 \\ 1 & . & . \\ 1 & . & . \end{pmatrix}, \quad \alpha\begin{pmatrix} . & 1 & -1 \\ 1 & . & . \\ -1 & . & . \end{pmatrix}, \quad \alpha\begin{pmatrix} . & -1 & 1 \\ -1 & . & . \\ 1 & . & . \end{pmatrix}, \quad \alpha\begin{pmatrix} . & -1 & -1 \\ -1 & . & . \\ -1 & . & . \end{pmatrix},$$

respectively.

These correspond to ellipsoids symmetrically placed about the axis of the array. Careful use of equation (4) verifies that such an assembly does indeed move according to equations (2)with a constant B given by

$$-\frac{1-p}{1+p}\left(1+\frac{\alpha(2p)^{\frac{1}{2}}}{1+p}\right),$$

where, as before, we have written $r^2/2l^2 = p$.

If α vanishes the array behaves like an ellipsoid of revolution with axis ratio $\sqrt{p}$, and continues to rotate indefinitely in a uniform simple shear. But if α is strictly greater than zero, however small, there are values of p which make $|B| > 1$. The condition for this is

$$p < \tfrac{1}{2}\alpha^2.$$

In this case the behaviour is completely different, and there are two orientations of stable equilibrium inclined at a small angle to the streamlines. In these positions the lateral migration does not vanish, but is

$$v = \tfrac{1}{2}\alpha G r.$$

In a slightly non-uniform shear the action of the walls in selecting particles drifting in one direction may result in a preferential migration to the centre, or to the walls, and in alignment upstream rather than downstream or vice versa.

The cessation of overturning if the array is very long and thin can also be predicted by considering the sign of the external couple required to prevent the array rotating when it is held with its axis parallel to the streamlines, but free to move laterally. The forces under such circumstances are illustrated in figure 4. There is a couple of magnitude $4AGr^2$ on the four ellipsoids which is balanced by that associated with the sideways force associated with the anisotropy of the resistance of the ellipsoids. If the array were not free to move laterally this force would be $4\alpha AGr$, but, because no net force can act on it, it actually migrates with speed $\frac{1}{2}\alpha Gr$ to first order in α. Taking moments about the point $(l, 0, 0)$ it is clear that no external couple will be required to maintain equilibrium in this position if

$$r/l = \alpha.$$

If r/l exceeds this value the transverse force is not sufficient to prevent continual overturning, but for smaller values of r/l the array tends to move away from alignment along the streamlines in the opposite sense. This movement must be checked before the axis of the array makes a substantial angle with the streamlines, but it is associated with the existence of an equilibrium orientation.

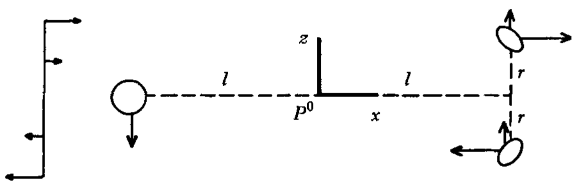

FIGURE 4. The forces on the array of ellipsoids described in § 5.1, when held with its axis parallel to the basic streamlines. It is moving sideways with velocity $\frac{1}{2}\alpha Gr$.

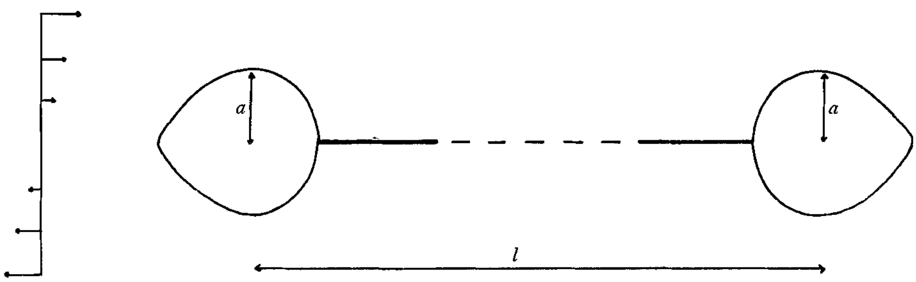

FIGURE 5. A long body of revolution for which $|B| > 1$, so that it has two opposite stable equilibrium orientations in a uniform simple shear.

5.3. In view of the importance of this effect it is of value to show that it may also exist in bodies strictly of revolution. Exact solutions for the velocity field round a body in a shear flow at low Reynolds number are known only for a sphere and an ellipsoid. If one assumes the body to be slender, one can obtain an approximate solution by a method similar to that of Hancock (1953), but this is unsatisfactory for a number of reasons and is omitted here. Another approach is to demonstrate the existence of a body of revolution on which, when held so that it cannot rotate in a simple shear with its axis parallel to the streamlines, there is a lateral force. If two such bodies are connected by a sufficiently long and thin rigid rod, as in figure 5, the constant B describing the combination will be numerically greater than unity, and the orbits will be quite different from those of Jeffery.

The simplest such body showing a lateral force is that with surface given in polar co-ordinates by

$$r = a\{1 + \alpha P_3(\mu)\}, \quad \alpha \ll 1,$$

where μ is the cosine of the angle made with Ox_1 by the radius vector, and $P_3(\mu)$ is the Legendre polynomial of degree 3. It is necessary for the degree of this polynomial to be odd, otherwise the lateral force vanishes by symmetry (as for an ellipsoid). The surface $r = a\{1+\alpha P_1(\mu)\}$ is merely a sphere displaced through a distance $a\alpha$ along Ox_1, so will also yield no lateral force. Straightforward, though tedious, algebra suffices to show that the velocity field round the body is

$$u_1 = G\left\{x_3\frac{\partial U}{\partial x_1}+x_1\frac{\partial^2 V}{\partial x_1\,\partial x_3}-\frac{\partial V}{\partial x_3}+\frac{\partial^2 W}{\partial x_1\,\partial x_3}+x_3\right\},$$

$$u_2 = G\left\{x_3\frac{\partial U}{\partial x_2}+x_1\frac{\partial^2 V}{\partial x_2\,\partial x_3}+\frac{\partial^2 W}{\partial x_2\,\partial x_3}\right\},$$

$$u_3 = G\left\{x_3\frac{\partial U}{\partial x_3}-U+x_1\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial^2 W}{\partial x_3^2}\right\},$$

where U, V, W, are harmonic functions and

$$U = \frac{1}{6}\frac{a^3}{r^2}\mu+\frac{\alpha}{252}\left\{27\frac{a^2}{r}+14\frac{a^4}{r^3}P_2+72\frac{a^6}{r^5}P_4\right\}+O(\alpha^2),$$

$$V = -\frac{4}{6}\frac{a^3}{r}+\frac{\alpha}{252}\left\{112\frac{a^4}{r^2}P_1-72\frac{a^6}{r^4}P_3\right\}+O(\alpha^2),$$

$$W = \frac{1}{6}\frac{a^5}{r^2}\mu+\frac{\alpha}{252}\left\{-113\frac{a^4}{r}+6\frac{a^6}{r^3}P_2+40\frac{a^8}{r^5}P_4\right\}+O(\alpha^2).$$

These velocities satisfy the Stokes equations, vanish on the body to first order in α, and represent a simple shear as $r \to \infty$. The only contribution to the couple on it comes from that part of the velocity field which varies as $1/r^2$, and is

$$4\pi\eta a^3 G+O(\alpha^2)$$

in the same direction as the vorticity. The force comes only from that part which varies as $1/r$, and is

$$\tfrac{6}{7}\pi\eta a^2\alpha G+O(\alpha^2)$$

parallel to Ox_3. Here η is the liquid viscosity. The minimum length of connexion for the value of $|B|$ for the combination to be greater than unity is around $28a/3\alpha$.

The particular body of revolution considered here is symmetrical between its two ends, and cannot be expected to show any tendency to lateral drift, even in the equilibrium orientation. But in general the total force on an asymmetrical body of revolution will only exactly vanish if it is moving sideways as a whole. This can be seen for the arrays of ellipsoids in figures 2 and 4. Such motion is a function of body shape and orientation alone, and if the orientation orbit is periodic and symmetrical the net displacement over a period must be zero. But if an equilibrium orientation is attained the motion will be systematic, and although slow will be of dominant importance.

However, it is very doubtful whether such a combination with $|B| > 1$ could in practice be constructed. The difficulty is that if the resistance of the connexion is to be negligible it must be extremely thin indeed. The resistance to lateral motion of a rod of length l and small radius d is of order $\eta l/\log(l/d)$. The velocity field round the almost spherical bodies at the ends has a component of order Ga^2/l perpendicular to the connexion, and in a direction assisting the overturning of the combination. This can only be neglected if

$$G\eta a^2 l/\log(l/d) \ll \tfrac{6}{7}\pi\eta a^2 l\alpha G-8\pi\eta a^3 G;$$

in other words, if $1/\log(l/d) \ll \alpha$. These are order of magnitude estimates, and it is not easy to calculate more precisely the multiplicative constants. A factor of 2 is of considerable importance, being associated with a ratio of 10 in the permissible thickness of the combination. The author's private guess is that this thickness must be too small to be consistent with rigidity.

5.4. This calculation underlines the unrealistic nature of the model of a general body which was used in §§ 3 and 4. The results of § 4 and of this section must be taken only as existence theorems about possible behaviour. They are undoubtedly extreme examples, probably of little practical importance. However, they are given to indicate the wide qualitative variation in the possible orbits of a general body, and as a warning against the assumption that all particles will move in orbits closely resembling those calculated by Jeffery.

6. Bodies of revolution in a three-dimensional uniform shear

6.1. It is perhaps of interest that a solution of the equations of motion for a body of revolution may be given for a uniform shear which is not simple, in which the basic flow is described by a constant three-dimensional rate of strain tensor **E**, and a rotation $\mathbf{\Omega}$. For simple shear **E** represents plane strain perpendicular to and numerically equal to the rotation. As shown in § 3, the tensor $B_{\alpha\beta\gamma}$ describing the body depends on one constant B, and the components of angular velocity relative to fixed axes which momentarily coincide with principal axes in the body are

$$\omega_1 = \Omega_1, \quad \omega_2 = \Omega_2 + Be_{31}, \quad \omega_3 = \Omega_3 - Be_{12}. \tag{5}$$

As the body rotates, the tensor $B_{\alpha\beta\gamma}$ loses this simple form but a simple transformation still enables the direction of the axis to be calculated. If x_α is a vector always in the direction of the axis of revolution (so that in the above co-ordinates $x_2 = x_3 = 0$ momentarily), but of length still to be specified, then

$$\dot{x}_2 = \omega_3 x_1 - \omega_1 x_3 = \Omega_3 x_1 - Be_{21} x_1,$$

and

$$\dot{x}_3 = \omega_1 x_2 - \omega_2 x_1 = -\Omega_2 x_1 - Be_{31} x_1.$$

We now *define* $\dot{x}_1$ by $\dot{x}_1 = -Be_{11}x_1$, so that all these three equations are summed up by

$$\dot{x}_\alpha = \epsilon_{\alpha\beta\gamma}\Omega_\beta x_\gamma - Be_{\alpha\gamma}x_\gamma. \tag{6}$$

This is derived only if at each moment the axes are rotated to coincide with some principal axes of the body. But it is a vector equation, so true in any co-ordinate frame, including that in which the components $\epsilon_{\alpha\beta\gamma}\Omega_\beta$, $e_{\alpha\gamma}$ of $\mathbf{\Omega}$, **E** appear as constants. The problem is thus reduced to a linear one, for which it is easy to write down the general solution.

We look for vectors $\hat{x}_\alpha$ satisfying for some σ,

$$(\epsilon_{\alpha\beta\gamma}\Omega_\beta - Be_{\alpha\gamma} - \sigma\delta_{\alpha\gamma})\hat{x}_\gamma = 0,$$

so that $x_\alpha = \hat{x}_\alpha e^{\sigma t}$ is a solution of equation (6). σ is then given by the roots of the determinantal equation

$$|\mathbf{\Omega} - B\mathbf{E} - \sigma\mathbf{I}| = 0.$$

This is an algebraic equation with real coefficients, so either all three roots are real, or one is real and the other two are complex conjugates. These cases will be considered separately.

If the eigenvalues are real, being $\lambda > \mu > \nu$, to each corresponds an eigenvector $\hat{L}_\alpha$, $\hat{M}_\alpha$, $\hat{N}_\alpha$. These are not necessarily orthogonal, as $\mathbf{\Omega} - B\mathbf{E}$ is not symmetric, but if the eigenvalues are distinct, they are linearly independent. The general solution to equation (6) is

$$x_\alpha = C\hat{L}_\alpha e^{\lambda t} + D\hat{M}_\alpha e^{\mu t} + E\hat{N}_\alpha e^{\nu t},$$

where C, D, E, are arbitrary constants. As $t \to \infty$, $x_\alpha \sim C\hat{L}_\alpha e^{\lambda t}$, because λ is the largest of the three roots, and the axis of the body tends to a position parallel to $\hat{L}_\alpha$. This, apart from the freak $C = 0$, is to within a change of sign independent of the initial position of the body. Two equal eigenvalues are also highly unlikely, as the eigenvalues depend on the value assigned to B, and thus on the individual particle considered.

If two eigenvalues are complex, $\xi \pm i\zeta$, say, with associated complex eigenvectors $\hat{X}_\alpha \pm i\hat{Z}_\alpha$, the general solution of equation (6) is

$$x_\alpha = C\hat{L}_\alpha e^{\lambda t} + \mathscr{R}\{(D+iE)(\hat{X}_\alpha + i\hat{Z}_\alpha) e^{(\xi + i\zeta)t}\}.$$

The behaviour as $t \to \infty$ depends on whether λ is greater than, equal to, or less than ξ. Because of the incompressible nature of the basic flow $\lambda + 2\xi = 0$. If $\lambda > 0$, $\xi < \lambda$ and $x_\alpha \sim C\hat{L}_\alpha e^{\lambda t}$ and the orientation tends to a definite direction. If, on the other hand, $\xi > 0$,

$$x_\alpha \sim \mathscr{R}\{(D+iE)(\hat{X}_\alpha + i\hat{Z}_\alpha) e^{(\xi+i\zeta)t}\},$$

and the axis rotates in a plane generated by $\hat{X}_\alpha$, $\hat{Z}_\alpha$ with period $2\pi/\zeta$. This behaviour is quite different from that calculated by Jeffery in a simple shear, for here there is one asymptotic plane in which the orbit lies.

Simple shear is a special case. Then both the rate of strain and rotation matrices $\mathbf{E}$, $\mathbf{\Omega}$ are singular, with the same null space. So $B\mathbf{E} - \mathbf{\Omega}$ is always singular, and the characteristic equation has at least one zero root. The rate of strain is numerically equal to the angular velocity, so the condition for real roots is $|B| > 1$ and one of these must be positive as the sum of the three roots vanishes. As before there is an asymptotic orientation, as was pointed out in the previous section. More usually, the roots are complex, $0, i\zeta, -i\zeta$, say. The general solution is then

$$x_\alpha = C\hat{L}_\alpha + D\{\hat{X}_\alpha \cos(\zeta t + \gamma) - \hat{Z}_\alpha \sin(\zeta t + \gamma)\}.$$

As $t \to \infty$ neither term dominates, and the orbit is always dependent on the initial conditions. As it is the *direction* of x_α which is important, except for the phase factor γ there is a singly infinite family of possible orbits determined by the ratio D/C.

A sharp distinction must thus be drawn between shear flows in which the motion near a point is locally a simple shear, and those where it has a slight three-dimensional character. The first case is probably more important, as it includes the Couette viscometer, and the flow down any parallel-sided pipe. In a diverging pipe, on the other hand, the three-dimensional character of the flow may be vital, at least if the particle remains in it long enough. For it will determine the ultimate orientation independently of its initial value, in complete contrast to what happens in a simple shear. Small imperfections in a locally two-dimensional flow will not necessarily result in a drift through the orbits calculated by Jeffery. Only if an imperfection moves down the streamlines at the same speed as the particle will it act for long enough to produce a systematic reorientation. Otherwise only small random drifts which are outside the scope of this analysis will result. Whether, in mean, these drifts would be exactly self-cancelling is a difficult question.

Equation (6) is identical to that describing the motion of one end of a line element of fluid with the other end at the origin under a rate of strain tensor $B\mathrm{E}$, and rotation $\boldsymbol{\Omega}$. The magnitude of the constant B thus measures the effectiveness of the rate of strain tensor in rotating the axis of symmetry, compared to a line element with the same orientation. Rotations about the axis do not affect the motion of the axis, and have not been calculated at all. For more general bodies this simplification is impossible, and the reduction to a linear problem cannot be effected in this way.

7. Conclusion

It is worth recapitulating the somewhat complicated argument of this paper. A single rigid particle is moving under no external forces in a unidirectional viscous flow field. If the particle is sufficiently small it will be carried with the local fluid velocity, and its changes of orientation will be determined by the local rates of strain and rotation, and may be calculated on the basis of the Stokes equations of motion. Using only the linearity and invariance under co-ordinate transformations of these equations it was shown in § 3.1 that the detailed shape of the particle affects its rate of rotation only through a third-rank tensor $B_{\alpha\beta\gamma}$, symmetric between the second and third suffices. In § 3.4, particles were found for which the coefficients $B_{\alpha\beta\gamma}$ assume arbitrarily assigned values, though for a particular choice of axes some are much smaller than others. For a body of revolution, however, symmetry properties enable us to deduce that only one arbitrary constant B is involved in the specification of $B_{\alpha\beta\gamma}$ (§ 3.2), and the orientation orbit may be solved completely. If $|B| < 1$, the ends of the axis of symmetry rotate periodically on one of the singly infinite family of spherical ellipses calculated by Jeffery (1923) for an ellipsoid of revolution. If $|B| > 1$, the orbit is of quite different character, and the axis moves into one of two oppositely directed orientations, and there remains. In § 5.3 an example is given of a body of revolution for which $|B| > 1$. It is, however, extremely long and thin, and could probably not be constructed of known materials. The author suspects this may be true of all such bodies. In § 4 another example is given of a body which tends to remain in particular orientations rather than repeatedly overturning. This is not a body of revolution and the couple which halts overturning arises in a different way.

In general a small particle will move laterally across the streamlines of a simple shear even if the Reynolds number is so small that inertia forces are negligible. The speed of migration will depend on the shape and orientation of the particle, but will be proportional to its size. However, it was shown in § 2 that for a rigid body of revolution if the orientation orbit is periodic lateral movement in one half of the cycle is balanced by an equal and opposite movement in the other. The same of true in any unidirectional flow bounded by rigid walls even if the particle is not small, provided only the character of the orientation orbit is such that the axis of symmetry lies in the plane perpendicular to the direction of flow at least twice. The orbit of any body of revolution is then strictly periodic and no systematic lateral migration takes place. If, however, $|B| > 1$ and the particle is fairly small, everywhere except possibly very near the walls or in a region of vanishing velocity gradient, the orientation orbit will not be characterized by repeated reversals but one of two orientations will be selected, and provided the particle is not symmetrical between its two ends systematic lateral migration will take place.

All these results assume single rigid particles moving under no external forces in a unidirectional flow when inertial effects are negligible. That rods and disks follow Jeffery's orientation orbits in a simple shear away from a wall has been verified by Trevelyan & Mason (1951), and that they do not move towards or away from a rigid wall by Goldsmith & Mason (1961*b*). Flexible disks, on the other hand, move towards the centre of a Poiseuille flow.

The same theoretical results hold if the flow is not strictly unidirectional, but is nevertheless invariant under translations along every streamline (as in a Couette viscometer). Also the conclusion that the orbit of a body of revolution of finite size is strictly periodic (except for translation down the streamlines) provided only successive reversals occur, is unaffected by a constant external force on the body parallel to the streamlines (as in a vertically mounted tube when there is imperfect density matching between the body and fluid). However, the character of the orientation orbit may be completely altered by such a force, and reversals inhibited. This would occur if one end of a long rod were much denser than the other. For a homogeneous sphere any axis is an axis of symmetry, and it is not necessary to postulate repeated reversals to rule out migration. The horizontal migration of a sphere sedimenting in a parallel-sided vertical tube which was observed by Christopherson & Dowson (1959) cannot be explained within the context of the Stokes equations. Systematic lateral migration may also be ruled out theoretically for an isolated doublet of two separate spheres in a unidirectional flow, for such a system has an axis of symmetry. If one sphere drifting down the streamlines is to overtake another one the line joining their centres must not be perpendicular to the streamlines more than once, else it must also be so at regular intervals indefinitely, and the spheres have rotated about one another at all times in the past and will continue to do so in the future.

The orbit of an extremely small particle in a general unidirectional flow must be described to a first approximation by the Stokes equations even though the overall Reynolds number may not be small. The question then arises whether neglected inertial forces might not systematically modify the orbits, in particular giving rise to a lateral drift. Such a drift would almost certainly be proportional to a higher power of the particle size than the first (the measurements of Segré & Silberberg 1961 suggest the third or fourth power) and would be masked by any drift calculated on the basis of the Stokes equations, which on dimensional grounds must vary linearly with particle size. However, it has been shown that for most bodies of revolution (including spheres and ellipsoids) the first-order drift vanishes, so the problem of computing that due to inertia is important.

In § 6 of this paper the orientation orbit of a rigid infinitesimal body of revolution is calculated when it is in a low Reynolds number flow which is not unidirectional, but geometrically similar at all points along a streamline with a varying scale. This would be true for low Reynolds number laminar flow in a conical pipe. The rate of strain and rotation tensors describing the local velocity gradients round a particle are unchanged in structure as it moves down a streamline, though their overall magnitude may alter. With suitable redefinition of the parameter describing time, the equations of motion are those of a particle in a constant infinite uniform shear which is three-dimensional in character. In general any body of revolution in such a flow will either move into one of two opposite orientations, or rotate until its axis lies in a particular plane in which it reverses itself periodically. These orbits are qualitatively distinct from the spherical ellipses calculated by Jeffery for particles with $|B| < 1$ in a simple shear. He found an infinite family of possible orbits, of which all except two are three-dimensional, and which was followed depended on the initial conditions. Here there is only one asymptotic orbit for almost all initial orientations, and it is either plane, or consists of two discrete oppositely directed orientations. Under these circumstances systematic migration of bodies of revolution with $|B| < 1$ which are not symmetrical between the two ends will frequently appear. Whereas, when the flow is unidirectional the local shear is a simple one, the orbits are periodic and the first-order migration vanishes.

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